Sit no. ______ ID: ____________________ Name: solutions

Calculus 2 Midterm Exam PM:1:10 - 3:10, 3 Aug., 2016

All electronic devices are OFF. Don't cheat in Exam, otherwise you will deserve a zero .
Any answer you produce must be supported with sufficient reasons.

1. (8%) The Cartesian coordinates $(-1,-\sqrt{3})$ and $(-2\sqrt{3},2)$.
   (a) Find polar coordinates $[r,\theta]$, where $r>0$ and $0\le\theta<2\pi$.
   (b) Find polar coordinates $[r,\theta]$, where $r<0$ and $0\le\theta<2\pi$.

$r=\sqrt{1+3}=2$ $\quad \cos\theta=-\frac{1}{2}$, $\sin\theta=-\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{4}{3}\pi$

2分 2分

$(-1,-\sqrt{3})=[2,\frac{4}{3}\pi]=[-2,\frac{\pi}{3}]$ ※

r=2

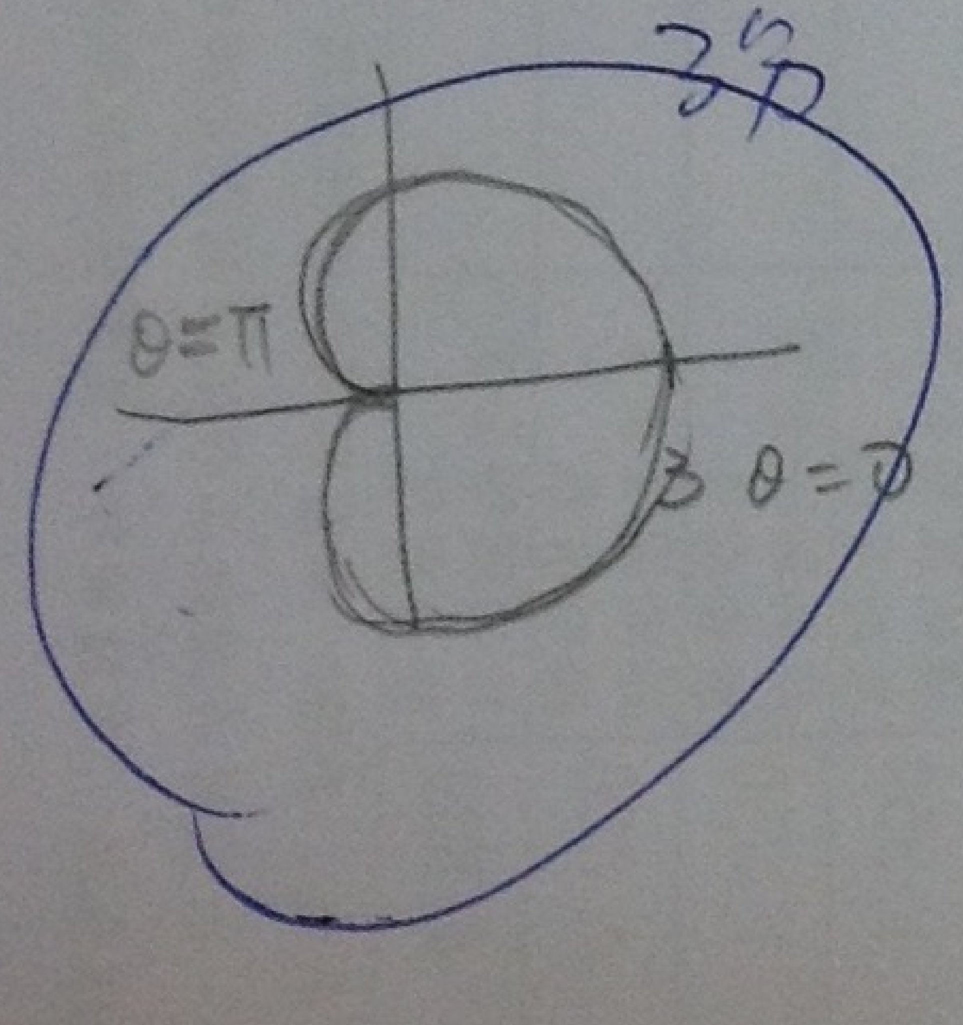
1分 r=4

$r=2\times2=4$ $\quad \cos\theta=-\frac{\sqrt{3}}{2}$, $\sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{5}{6}\pi$

2分 2分

$(-2\sqrt{3},2)=[4,\frac{5}{6}\pi]=[-4,\frac{11}{6}\pi]$ ※

2. 10 ~~(7%)~~ sketch and Find the arc length of cardioid $r=\frac{3}{2}+\frac{3}{2}\cos\theta$.



$ds=\sqrt{r^2+\dot{r}^2}\,d\theta$

$(\dot{r})^2=(-\frac{3}{2}\sin\theta)^2$

$+\quad r^2=(\frac{3}{2}+\frac{3}{2}\cos\theta)^2$

$r^2+\dot{r}^2=\frac{9}{4}+\frac{9}{4}+\frac{9}{2}\cos\theta=\frac{9}{2}(1+\cos\theta)$ +3

$L=2\int_0^{\pi}3\sqrt{\frac{1+\cos\theta}{2}}\,d\theta$ (+4) $=6\int_0^{\pi}\cos\frac{\theta}{2}\,d\frac{\theta}{2}\cdot 2$ $\quad \boxed{\cos^2\frac{\theta}{2}=\frac{1+\cos\theta}{2}}$

不合[illegible] −3

$=12\sin\frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi}=12$ ※

3. (15%) The parametric curve $C: x = t^3 - 3t,\ y = t^2$.
(a) Show that C has two tangents at $(0,3)$ and find their equations,
(b) Find the points on C where tangent is horizontal or vertical,
(c) Determine where the curve is concave upward or downward,
(d) Sketch the curve and (e) Find the area of the loop of C.

(a) $\begin{cases} t^3 - 3t = 0 \\ t^2 = 3 \end{cases} \Rightarrow t = \pm\sqrt{3}$, $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{2t}{3t^2-3}\Big|_{t=\pm\sqrt{3}} = \dfrac{\pm\sqrt{3}}{3}\left(\text{or } \pm\dfrac{1}{\sqrt{3}}\right)$

eq: $\underline{y = \pm\frac{1}{\sqrt{3}}x + 3}$ ※

(b) $\dot{y} = 0 \Rightarrow t = 0$ ∴ $\underline{(0,0) \text{ is H.T pt}}$ ※

$\dot{x} = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$, $t = 1$ $\underline{(-2,1)}$ are V.T pts
$t = -1$ $\underline{(2,1)}$ ※

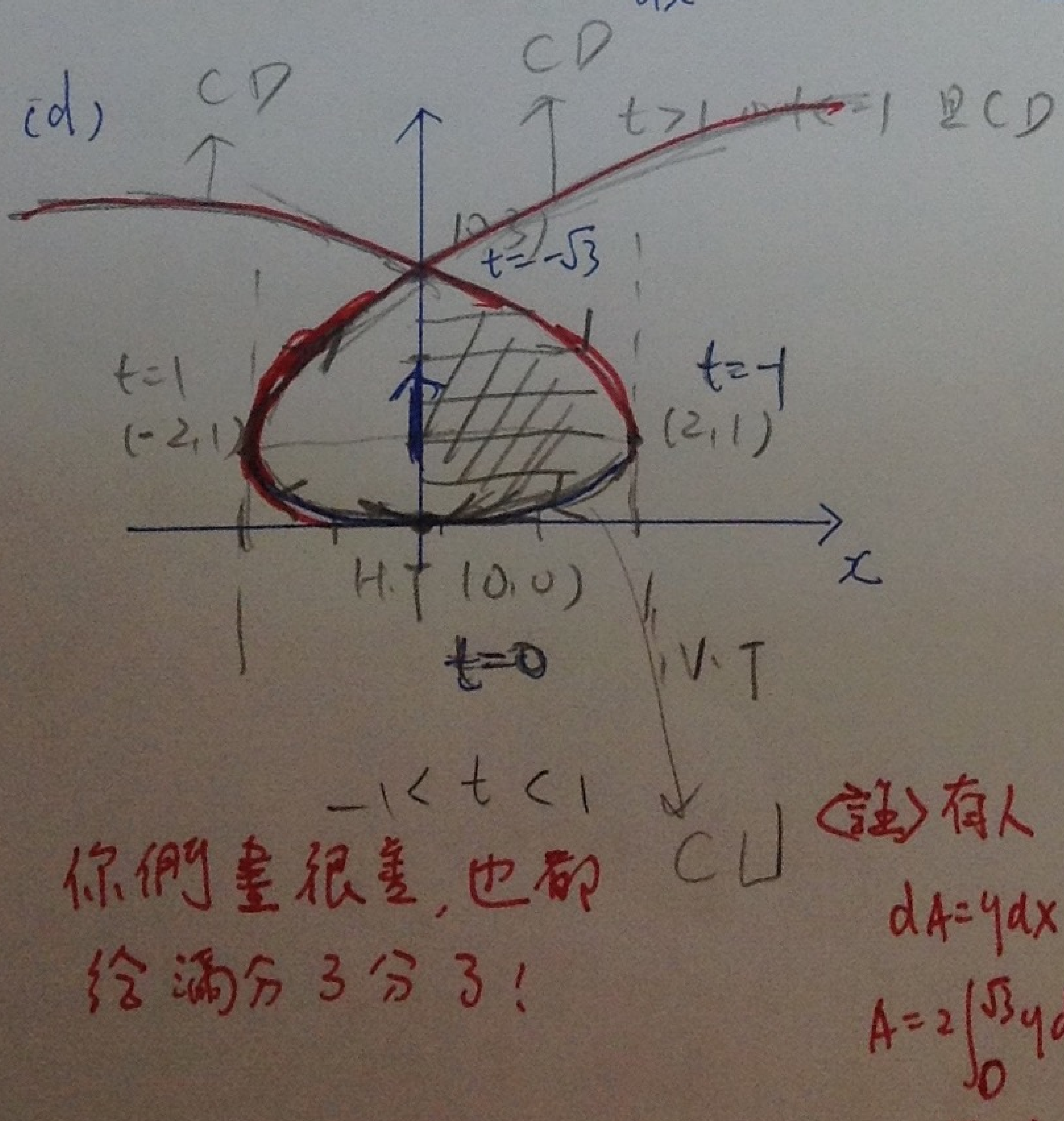
(c) $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \dfrac{\frac{2}{3}\frac{-t^2-1}{(t^2-1)^2}}{3(t^2-1)} = -\dfrac{2}{9}\dfrac{(t^2+1)}{(t^2-1)^3} = -\dfrac{2}{9}\dfrac{\overset{>0}{(t^2+1)}}{\underset{>0}{(t^2-1)^2}(t^2-1)}$

解不等式,有些人有問題喔!

If $t^2-1 < 0$ then $\dfrac{d^2y}{dx^2} > 0$, 因此 $\underline{-1 < t < 1, \text{是 CU}}$ ※

If $t^2-1 > 0$ then $\dfrac{d^2y}{dx^2} < 0$, 因此 $\underline{t > 1 \text{ or } t < -1, \text{是 CD}}$ ※

(d)



你們畫很差,也都给滿分3分了!

(e) $t: 0 \to -\sqrt{3}$

$dA = \underline{x\,dy}$ 不含已

$A = 2\displaystyle\int_0^{-\sqrt{3}} \underbrace{(t^3-3t)}_{x}\underbrace{2t\,dt}_{dy}$

$= 4\displaystyle\int_0^{-\sqrt{3}} (t^4 - 3t^2)\,dt$

〈註〉有人
$dA = y\,dx$
$A = 2\displaystyle\int_0^{\sqrt{3}} y\,dx$
也算对!
我相信你是歪打正着

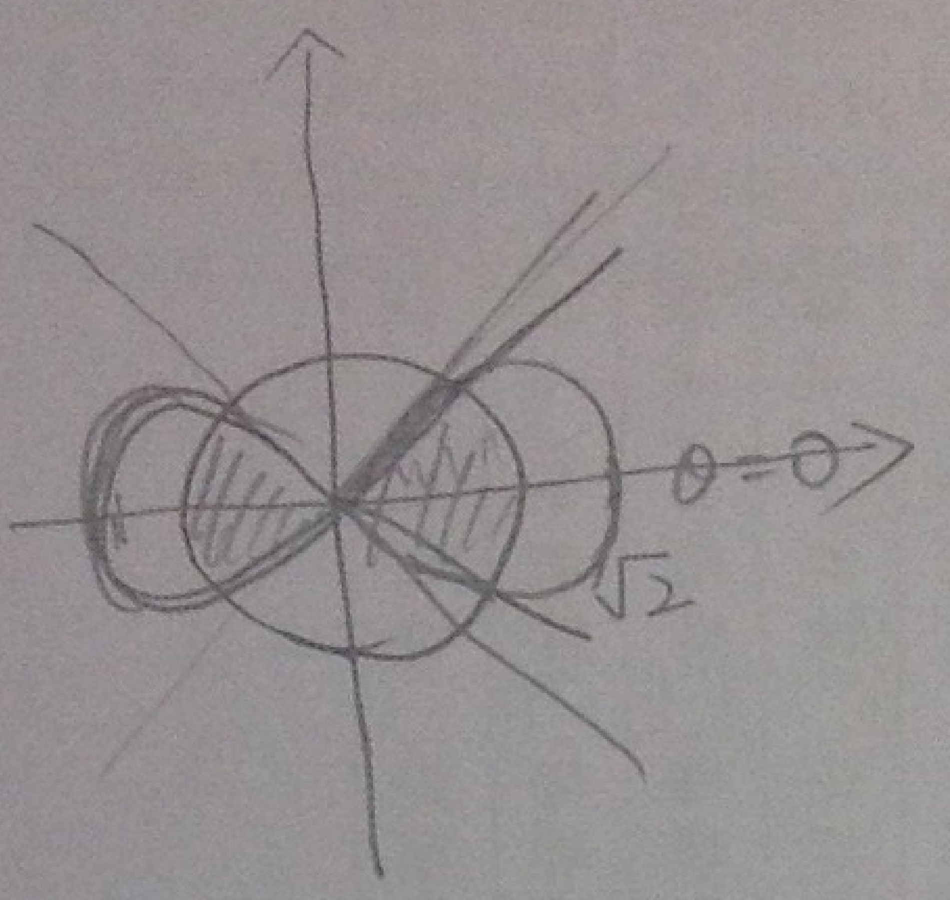
$= 4\left[\frac{1}{5}t^5 - t^3\right]\Big|_0^{-\sqrt{3}}$

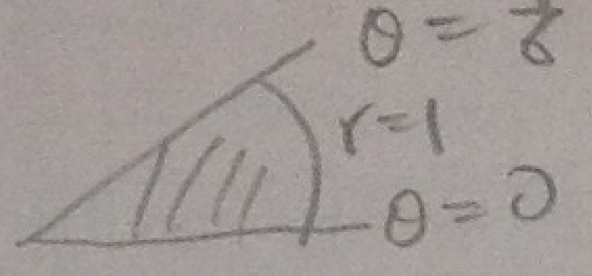
$= \underline{\dfrac{24}{5}\sqrt{3}}$ ※

4. (10%) Sketch curves $r^2 = 2\cos 2\theta$ and $r = 1$ and find the area of region that lies inside both curves.

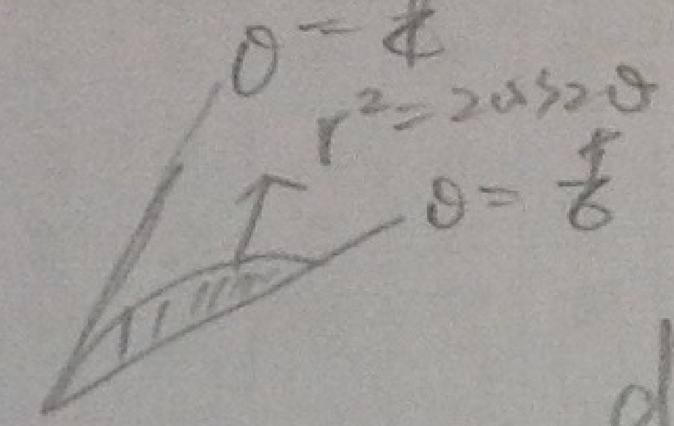
10 sketch and

4. ($\cancel{8}$%) Find the area of region that lies inside both curves $\boxed{r^2 = 2\cos 2\theta}$ and $r = 1$.





$\Rightarrow$ area $= \frac{1}{2} \times 1^2 \times \frac{\pi}{6} = \frac{\pi}{12}$



$dA = \frac{1}{2} r^2 d\theta$

$\Downarrow$

$$\text{area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\sin 2\theta \Big|_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{4}}\right) =$$

$$= \frac{1}{2}\left[1 - \frac{\sqrt{3}}{2}\right] = \frac{2-\sqrt{3}}{4}$$

$$\begin{cases} r^2 = 2\cos 2\theta \\ r = 1 \end{cases}$$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}$

THus, 面積 $= 4\left[\frac{\pi}{12} + \frac{2-\sqrt{3}}{4}\right] = \underline{2 - \sqrt{3} + \frac{\pi}{3}}$ #

5. (10%) Let $\vec{u} = <-2,1,4>$, $\vec{v} = <-3,5,2>$ and $\vec{w} = <1,-7,6>$, Then

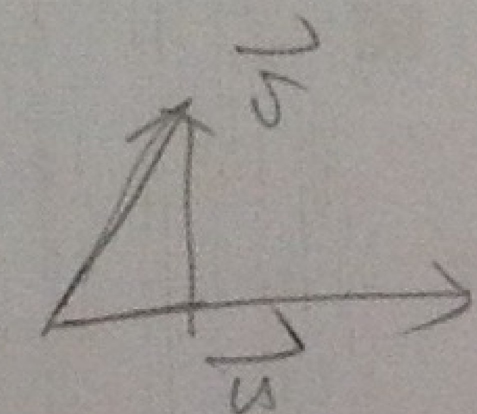
(a) $\vec{u}\cdot(\vec{v}\times\vec{w}) =$ <u>−4</u>, (b) $(\vec{v}\times\vec{u})\cdot\vec{w} =$ <u>4</u>, (c) $(\vec{v}\times\vec{u})\cdot\vec{u} =$ <u>0</u>,

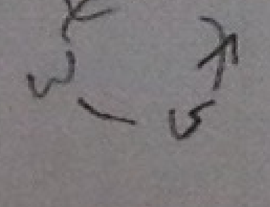
(d) The volume of the parallelepiped determined by $\vec{u}, \vec{v}$ and $\vec{w}$= <u>4</u>,

(e) $\text{proj}_{\vec{u}}\vec{v}$= $\frac{19}{21}<-2,1,4>$.

(a) $\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} -2 & 1 & 4 \\ -3 & 5 & 2 \\ 1 & -7 & 6 \end{vmatrix} = -60 + 84 + 2 - 20 - 28 + 18$

$= \underline{-4}$ #



(b)  $(\vec{v}\times\vec{u})\cdot\vec{w} = \underline{4}$ #

(c) $(\vec{v}\times\vec{u})\cdot\vec{u} = \underline{0}$ #

(d) Vol $= |-4| = \underline{4}$ #

(e) $\text{comp}_{\vec{u}}\vec{v}$

$= \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|} = \frac{\cancel{6+5+8}\ 19}{\sqrt{21}}$

$\text{proj}_{\vec{u}}\vec{v} = \frac{19}{\sqrt{21}} \frac{1}{\sqrt{21}} <-2,1,4>$

$= \underline{\frac{19}{21}<-2,1,4>}$ #

or $= <\frac{-38}{21}, \frac{19}{21}, \frac{76}{21}>$

3x5

6. (15%) (a) Find the equation of the plane E through $A(0,0,1)$, $B(2,3,-1)$, and $C(4,-3,0)$,
   (b) Find the volume of the solid enclosed by E and 3 coordinate planes,
   (c) Find the distance for the point $P(1,1,1)$ to the plane E,
   (d) Find the parametric equations of the line L through A and P.
   (e) Find the angle between L and E .

(a) (+1) $\begin{cases} \overrightarrow{AB} = \langle 2,3,-2 \rangle \\ \overrightarrow{AC} = \langle 4,-3,-1 \rangle \end{cases}$

(+1) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -2 \\ 4 & -3 & -1 \end{vmatrix} = \langle -9,-6,-18 \rangle$

$= -3 \underbrace{\langle 3,2,6 \rangle}_{= \vec{n}}$

(+1) $3x+2y+6z=6$

(b) $\frac{x}{2}+\frac{y}{3}+z=1$

$\frac{1}{3}\left(\underbrace{\frac{1}{2}\times(2\times3)}\times 1\right)=1$

4 錐 底面積×高

(c) $d(P,E) = \frac{5}{7} = \frac{|3+2+6-6|}{\sqrt{3^2+2^2+6^2}}$

(d) $\overrightarrow{AP} = \langle 1,1,0 \rangle$

$\begin{cases} x = t \\ y = t \\ z = 1 \end{cases}$

(e) $\vec{n} = \langle 3,2,6 \rangle$

$\vec{v} = \langle 1,1,0 \rangle$

$\cos\theta = \frac{3+2}{7\sqrt{2}} = \frac{5}{7\sqrt{2}}$ or $\frac{5\sqrt{2}}{14}$ $\quad \theta = \cos^{-1}\frac{5}{7\sqrt{2}}$

$\boxed{\frac{\pi}{2} - \cos^{-1}\frac{5}{7\sqrt{2}}}$ = 或 $\sin^{-1}\frac{5}{7\sqrt{2}}$

7. (5%) If C is a smooth curve defined by the vector function $\vec{r}(t)$.

Show the curvature $\kappa(t) = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$.

Pf: $\kappa(t) = \left|\dfrac{d\vec{T}(t)}{ds}\right| = \dfrac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$

Since unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, we have

$\vec{r}\,'(t) = |\vec{r}\,'(t)|\,\vec{T}(t)$

$\Rightarrow \vec{r}\,''(t) = \dfrac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{|\vec{r}\,'(t)|}\vec{T}(t) + |\vec{r}\,'(t)|\,\vec{T}'(t)$

$\Rightarrow \vec{r}\,'(t)\times\vec{r}\,''(t) = |\vec{r}\,'(t)|^2\,(\vec{T}(t)\times\vec{T}'(t))$ (since $\vec{T}\times\vec{T} = \vec{0}$)

$\Rightarrow |\vec{r}\,'(t)\times\vec{r}\,''(t)| = |\vec{r}\,'(t)|^2\,|\vec{T}'(t)|$ (since $|\vec{T}| = 1$ and $\vec{T}\perp\vec{T}'$)

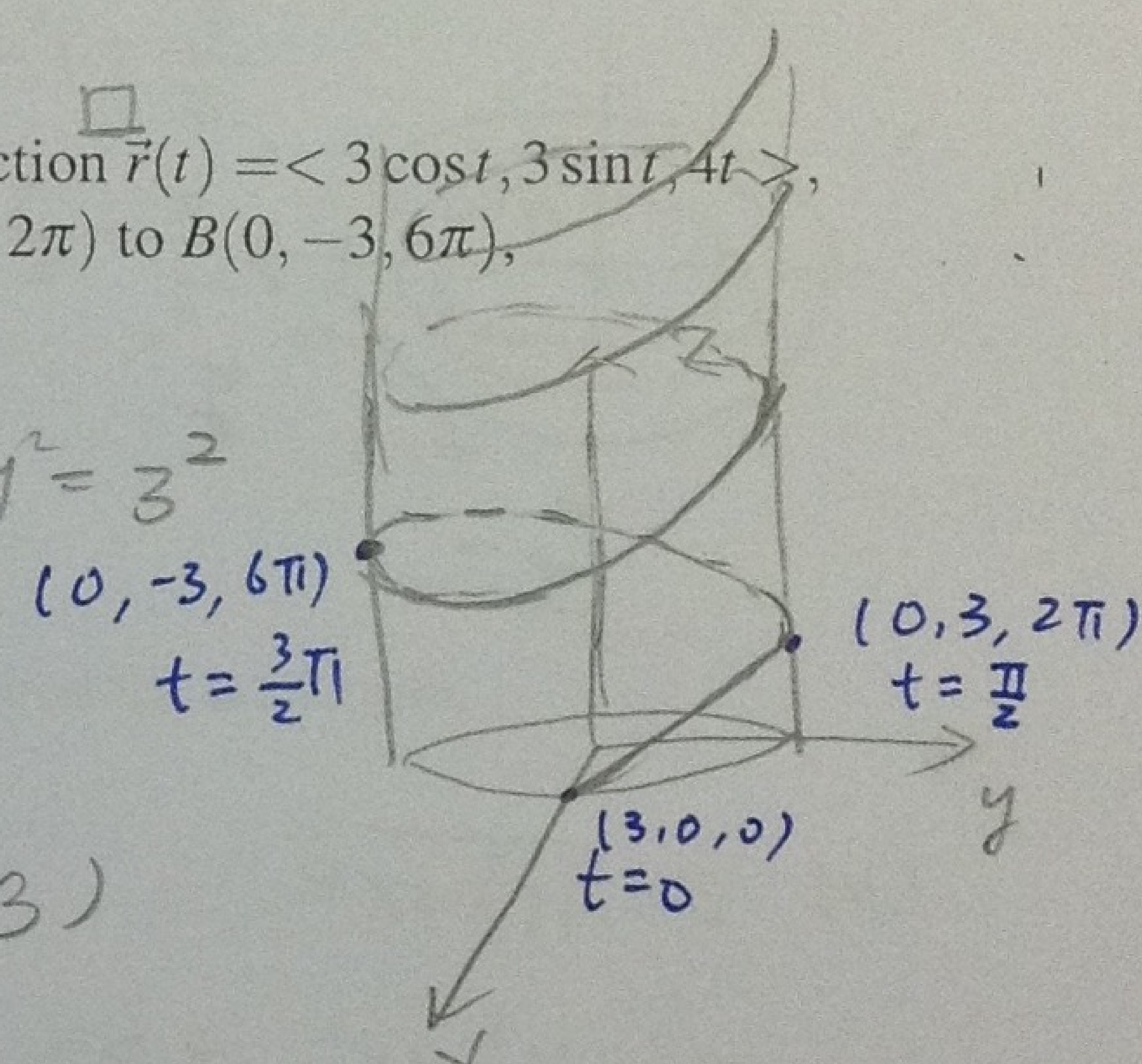
$\Rightarrow \kappa(t) = \dfrac{|\vec{T}'|}{|\vec{r}\,'|} = \dfrac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^3}$ $\square$

8. (12 %) (a) Sketch the helix with the vector function $\vec{r}(t) = < 3\cos t, 3\sin t, 4t >$,

(b) Find the arc length of the helix from $A(0,3,2\pi)$ to $B(0,-3,6\pi)$,

(c) Find the curvature $\kappa(t)$ of the helix $\vec{r}(t)$.

sol: (a) $\begin{cases} x = 3\cos t \\ y = 3\sin t \\ z = 4t \end{cases} \Rightarrow x^2 + y^2 = 3^2$



(b) $t: \frac{\pi}{2} \rightarrow \frac{3}{2}\pi$ $(A \rightarrow B)$

$\begin{cases} \dot{x} = -3\sin t \\ \dot{y} = 3\cos t \\ \dot{z} = 4 \end{cases}$ $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = 5$

$\underline{ds = 5\,dt}$

$L = \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} 5\,dt = 5\times\pi = \underline{5\pi}$ ✗

(c) $\vec{r}\,'(t) = < -3\sin t, 3\cos t, 4 > \Rightarrow |\vec{r}\,'| = 5$

$\vec{r}\,''(t) = < -3\cos t, -3\sin t, 0 >$

$\vec{r}\,'\times\vec{r}\,'' = < -12\sin t, -12\cos t, 9 >$

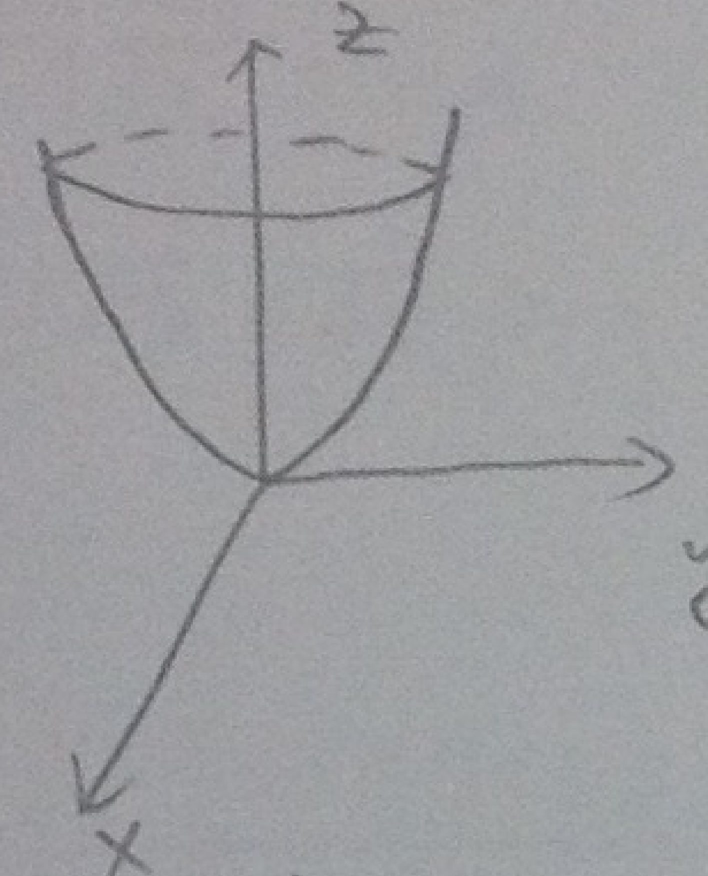
$\Rightarrow |\vec{r}\,'\times\vec{r}\,''| = 15$ and hence $\kappa(t) = \dfrac{15}{5^3} = \underline{\dfrac{3}{25}}$ ✗

$3\times5\%$ 9. (15%) Identify and sketch the graph of each surface and sketch its level curves for $k=0,1,2,3$. 1 2 2

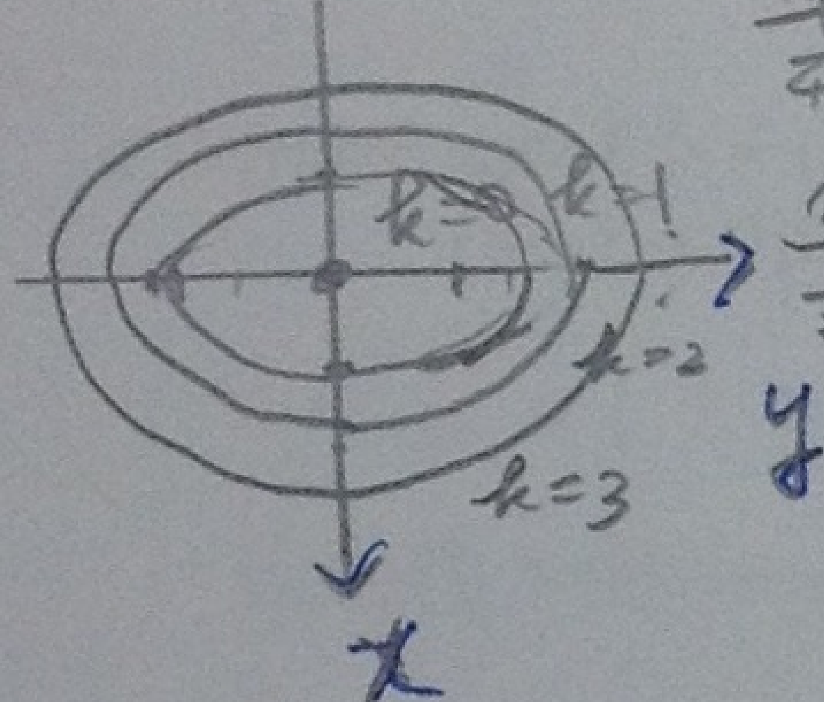
(a) $f(x,y)=y^2+4x^2$, (b) $g(x,y)=\sqrt{y^2+4x^2}$, (c) $h(x,y)=y^2-4x^2$.

(a) $z=4x^2+y^2$

- elliptic paraboloid



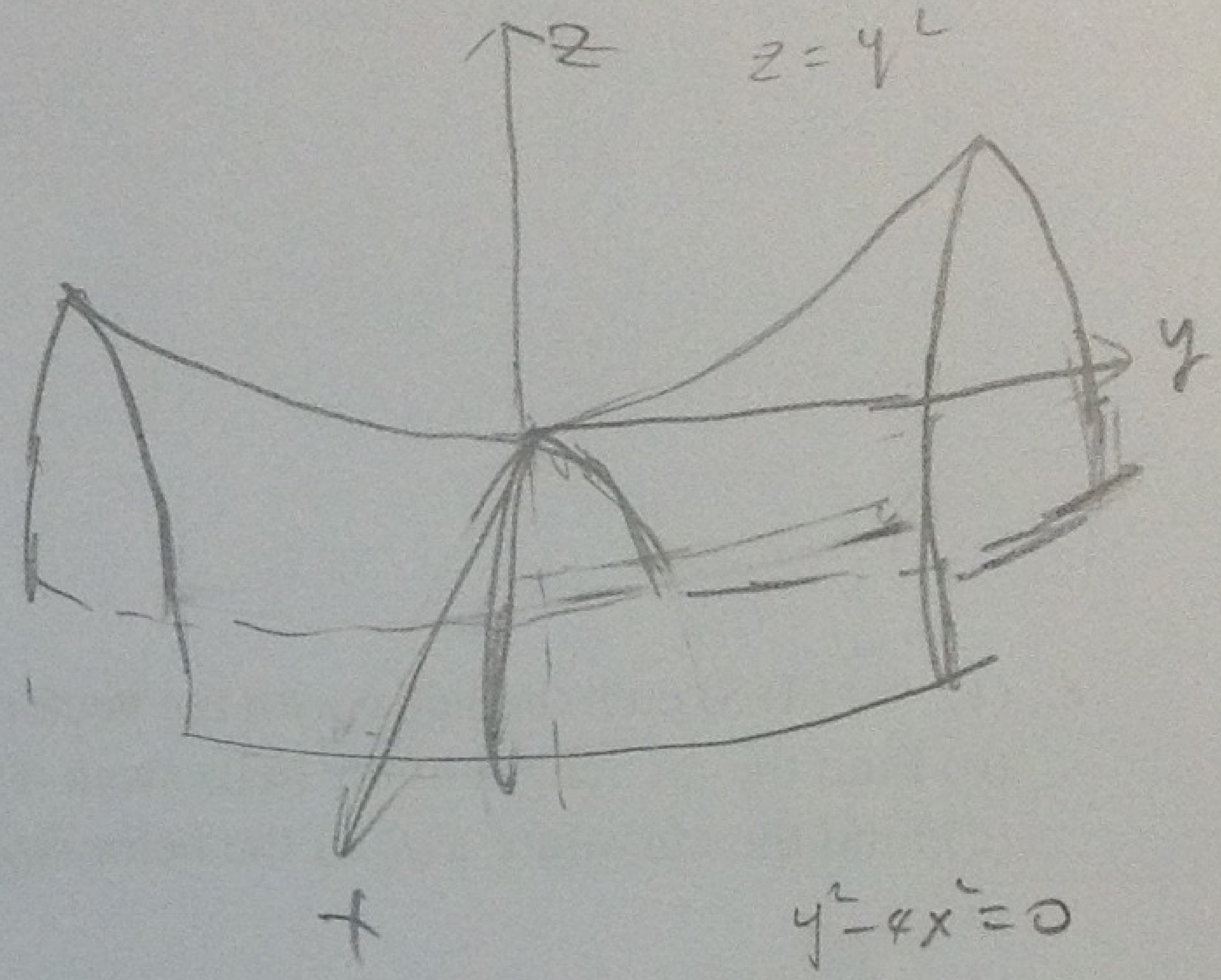
$\frac{x^2}{\frac{1}{4}}+y^2=1$



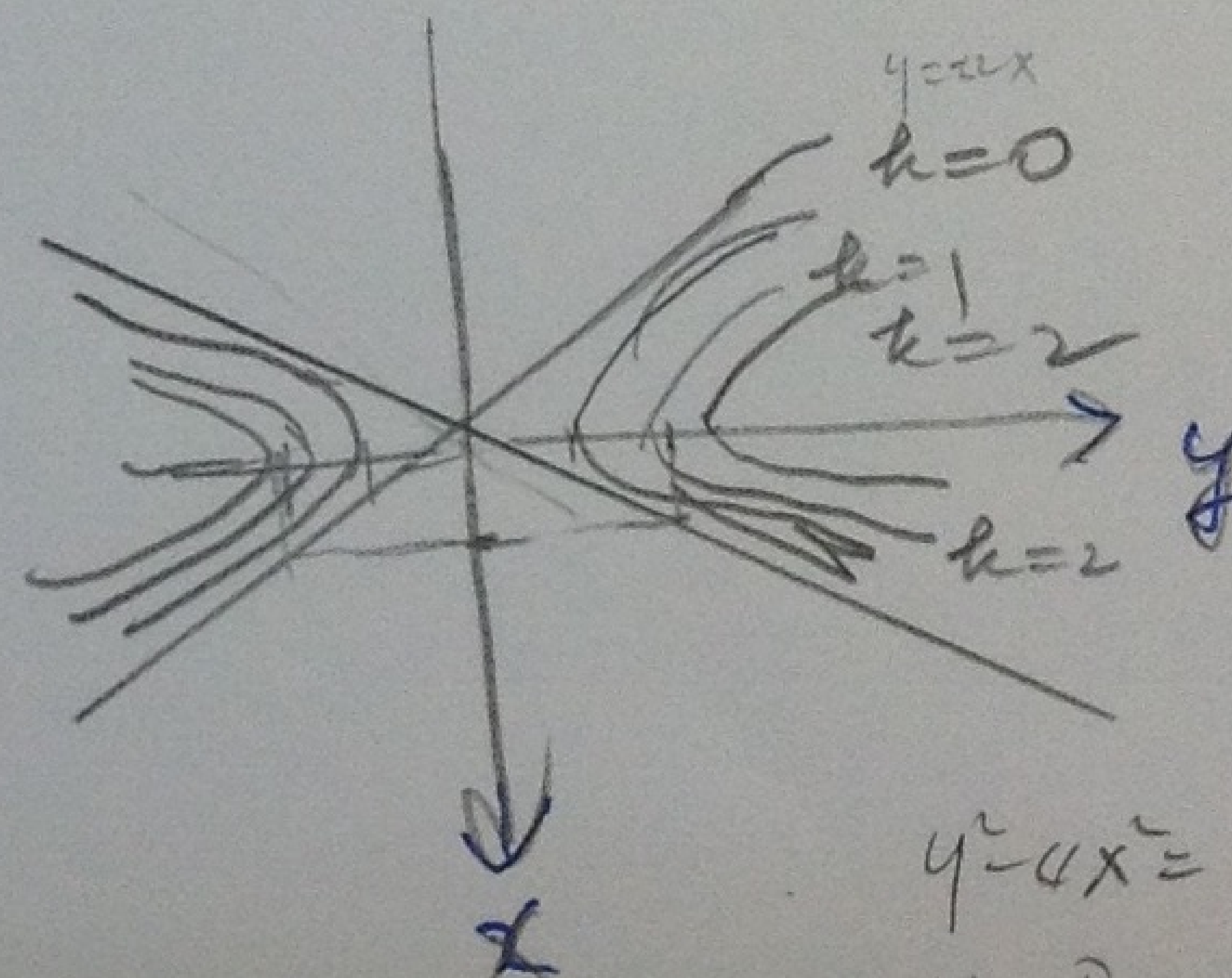
$\frac{x^2}{\frac{1}{2}}+\frac{y^2}{2}=1$

$z=y^2-4x^2$

hyperbolic paraboloid



$y^2-4x^2=0$



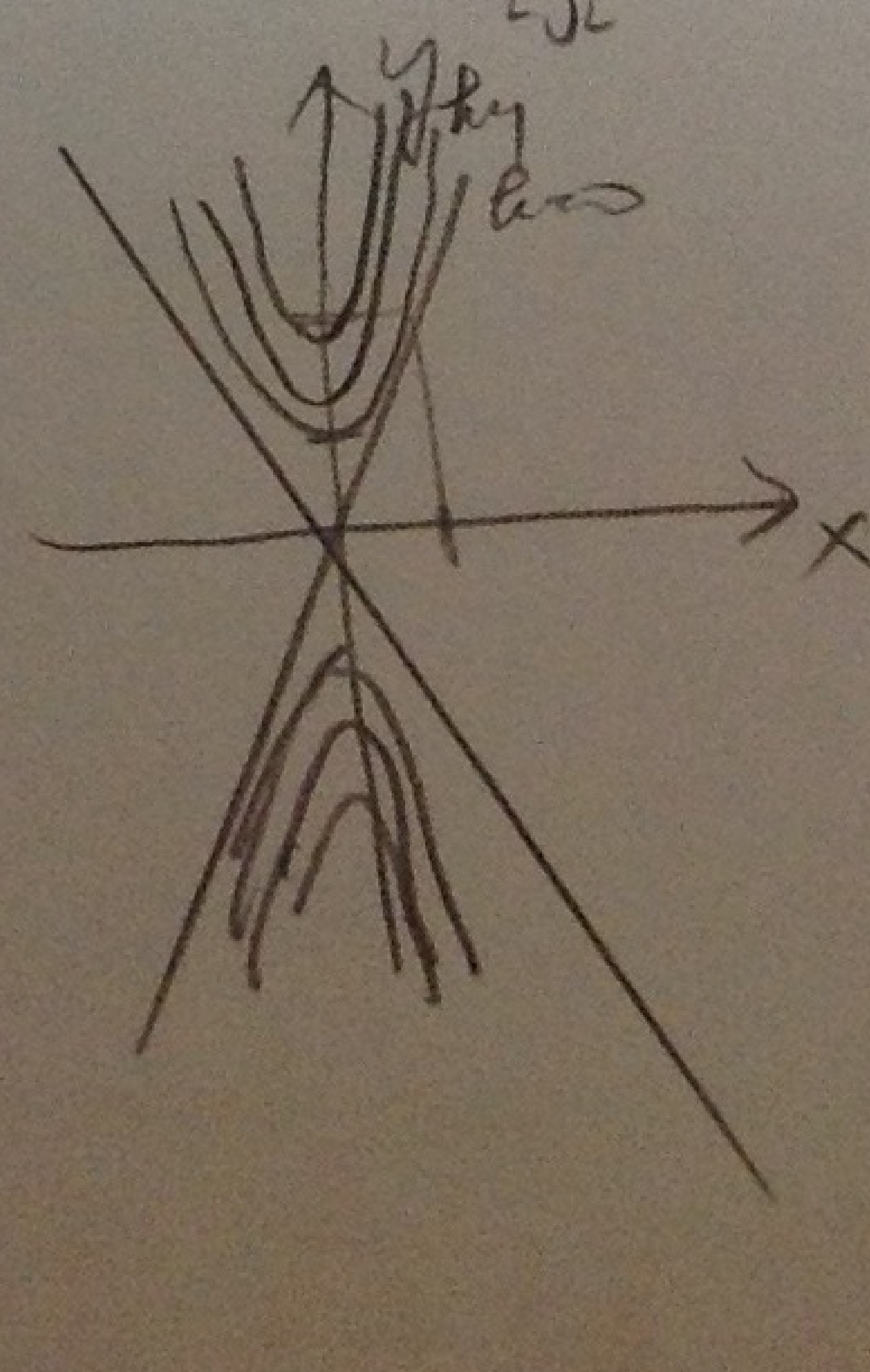
$y^2-4x^2=1$

±1 0

$y^2-4x^2=2$

$\pm\sqrt{2}$

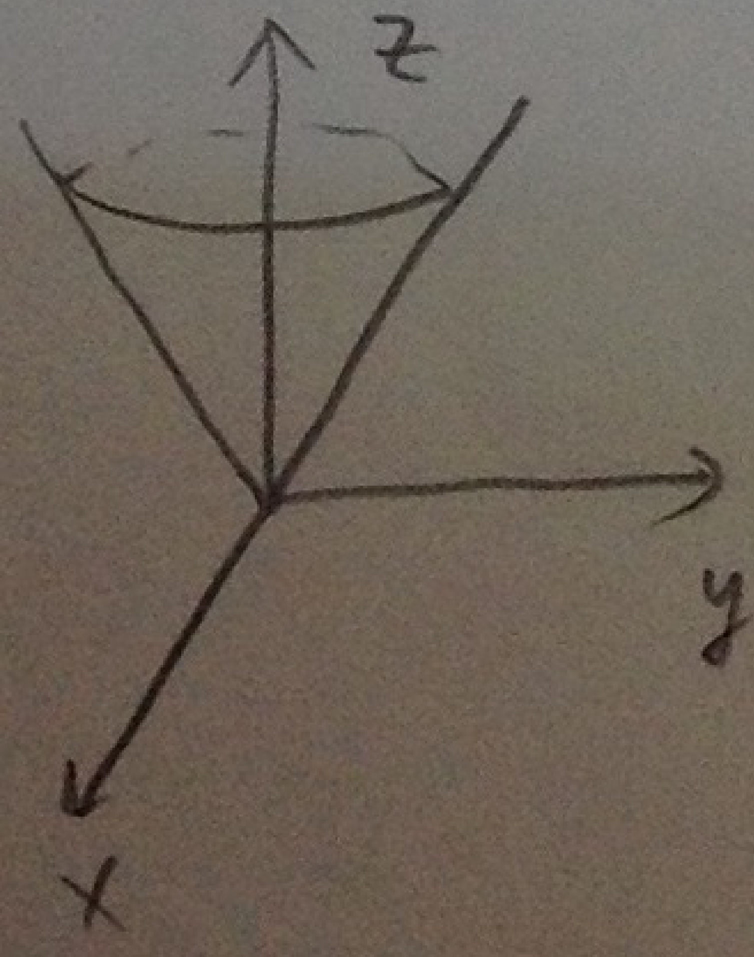
or

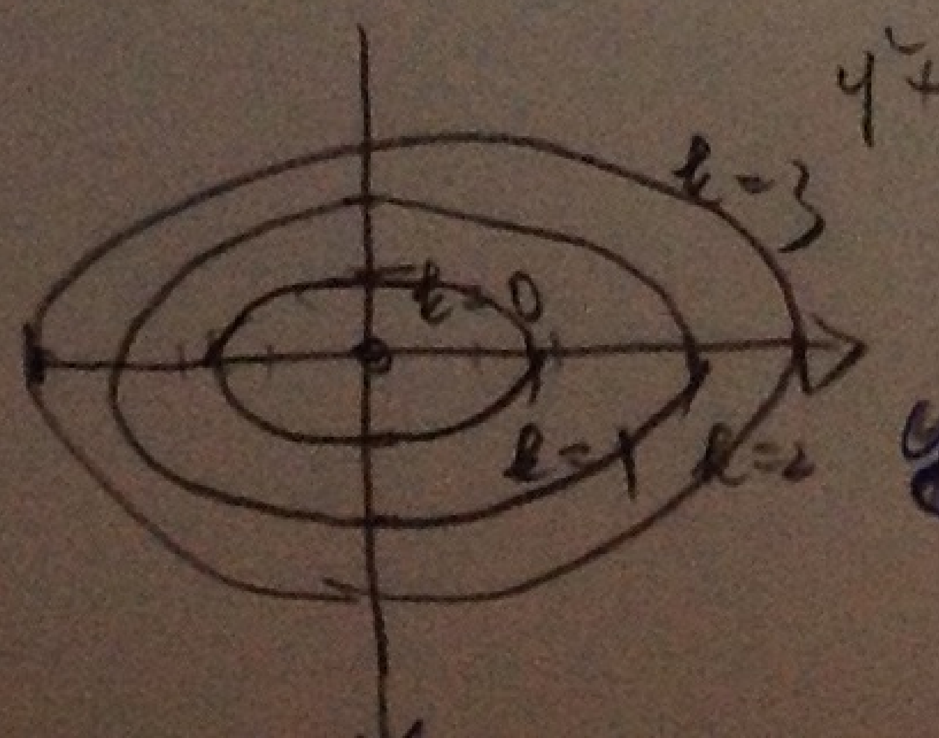


(b) $z=\sqrt{y^2+4x^2}$

$z^2=y^2+4x^2$, $z\geq0$

elliptic cone





$y^2+4x^2=1$ $\frac{x^2}{\frac{1}{4}}+\frac{y^2}{1}=1$

$y^2+4x^2=4$ $x^2+\frac{y^2}{4}=1$

$y^2+4x^2=9$ $\frac{y^2}{9}+\frac{x^2}{\frac{9}{4}}=1$